Analytical Solutions to the Navier-Stokes Equations

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Abstract

With the previous results for the analytical blowup solutions of the N-dimensional $(N \geq 2)$ Euler-Poisson equations, we extend the similar structure to construct an analytical family of solutions for the isothermal Navier-Stokes equations and pressureless Navier-Stokes equations with density-dependent viscosity.

1 Introduction

The Navier-Stokes equations can be formulated in the following form:

$$\left\{ \begin{array}{rl} \rho_t + \nabla\cdot(\rho u) & = 0, \\ (\rho u)_t + \nabla\cdot(\rho u\otimes u) + \delta\nabla P & = vis(\rho,u). \end{array} \right. \tag{1}$$

As usual, $\rho = \rho(x,t)$ and $u(x,t)$ are the density, the velocity respectively. $P = P(\rho)$ is the pressure. We use a γ-law on the pressure, i.e.

$$P(\rho) = K\rho^{\gamma}, \tag{2}$$

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with $K>0$, which is a universal hypothesis. The constant $\gamma=c_P/c_v\geq 1$, where c_p and c_v are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats. γ is the adiabatic exponent in (2). In particular, the fluid is called isothermal if $\gamma=1$. It can be used for constructing models with non-degenerate isothermal fluid. δ can be the constant 0 or 1. When $\delta=0$, we call the system is pressureless; when $\delta=1$, we call that it is with pressure. And $vis(\rho,u)$ is the viscosity function. When $vis(\rho,u)=0$, the system (1) becomes the Euler equations. For the detailed study of the Euler and Navier-Stokes equations, see [1] and [4]. In the first part of this article, we study the solutions of the N-dimensional ($N\geq 1$) isothermal equations in radial symmetry:

$$\begin{cases} \rho_t+u\rho_r+\rho u_r+\frac{N-1}{r}\rho u=0, \\ \rho\left(u_t+uu_r\right)+\nabla K\rho = vis(\rho,u). \end{cases} \tag{3}$$

Definition 1 (Blowup) *We say a solution blows up if one of the following conditions is satisfied:*
(1)The solution becomes infinitely large at some point x and some finite time T;
(2)The derivative of the solution becomes infinitely large at some point x and some finite time T.

For the formation of singularity in the 3-dimensioanl case for the Euler equations, please refer the paper of Sideris [10]. In this article, we extend the results form the study of the (blowup) analytical solutions in the N-dimensional ($N\geq 2$) Euler-Poisson equations, which describes the evolution of the gaseous stars in astrophysics [2], [3], [7], [12] and [13], to the Navier-Stokes equations. For the similar kinds of blowup results in the non-isothermal case of the Euler or Navier-Stokes equations, please refer [5] and [12].

Recently, Yuen's results in [13], there exists a family of the blowup solution for the Euler-Poisson equations in the 2-dimensional radial symmetry case,

$$\begin{cases} \rho_t+u\rho_r+\rho u_r+\frac{1}{r}\rho u=0, \\ \rho\left(u_t+uu_r\right)+K\rho_r=-\frac{2\pi\rho}{r}\int_0^r\rho(t,s)sds. \end{cases} \tag{4}$$

The solutions are

$$\begin{cases} \rho(t,r)=\frac{1}{a(t)^2}e^{y(r/a(t))},\ u(t,r)=\frac{\dot{a}(t)}{a(t)}r; \\ \ddot{a}(t)=-\frac{\lambda}{a(t)},\ a(0)=a_0>0,\ \dot{a}(0)=a_1; \\ \ddot{y}(x)+\frac{1}{x}\dot{y}(x)+\frac{2\pi}{K}e^{y(x)}=\mu,\ y(0)=\alpha,\ \dot{y}(0)=0, \end{cases} \tag{5}$$

where $K > 0$, $\mu = 2\lambda/K$ with a sufficiently small λ and α are constants.

(1)When $\lambda > 0$, the solutions blow up in a finite time T;

(2)When $\lambda = 0$, if $a_1 < 0$, the solutions blow up at $t = -a_0/a_1$.

In this paper, we extend the above result to the isothermal Navier-Stokes equations in radial symmetry with the usual viscous function

$$vis(\rho, u) = v\Delta u,$$

where v is a positive constant:

$$\begin{cases} \rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \\ \rho\left(u_t + uu_r\right) + K\rho_r = v(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u), \end{cases} \tag{6}$$

Theorem 2 *For the N-dimensional isothermal Navier-Stokes equations in radial symmetry (6), there exists a family of solutions, those are:*

$$\begin{cases} \rho(t,r) = \frac{1}{a(t)^N}e^{y(r/a(t))}, u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \\ \ddot{a}(t) = \frac{-\lambda}{a(t)}, a(0) = a_0 > 0, \dot{a}(0) = a_1, \\ y(x) = \frac{\lambda}{2K}x^2 + \alpha, \end{cases} \tag{7}$$

where α and λ are arbitrary constants.

In particular, for $\lambda > 0$, the solutions blow up in finite time T.

In the last part, the corresponding solutions to the pressureless Navier-Stokes equations with density-dependent viscosity is also studied.

2 The Isothermal $(\gamma = 1)$ Cases

Before we present the proof of Theorem 2, the Lemma 6 of [13] could be needed to further extended to the N-dimensional space.

Lemma 3 (The Extension of Lemma 6 of [13]) *For the equation of conservation of mass in radial symmetry:*

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \tag{8}$$

there exist solutions,

$$\rho(t,r)=\frac{f(r/a(t))}{a(t)^N},\ \ u(t,r)=\frac{\dot{a}(t)}{a(t)}r, \tag{9}$$

with the form $f\geq 0\in C^1$ *and* $a(t)>0\in C^1$.

Proof. We just plug (9) into (8). Then

$$\begin{aligned}
&\rho_t+u\rho_r+\rho u_r+\frac{N-1}{r}\rho u\\
&=\frac{-N\dot{a}(t)f(r/a(t))}{a(t)^{N+1}}-\frac{\dot{a}(t)r\dot{f}(r/a(t))}{a(t)^{N+2}}\\
&+\frac{\dot{a}(t)r}{a(t)}\frac{\dot{f}(r/a(t))}{a(t)^{N+1}}+\frac{f(r/a(t))}{a(t)^N}\frac{\dot{a}(t)}{a(t)}+\frac{N-1}{r}\frac{f(r/a(t))}{a(t)^N}\frac{\dot{a}(t)}{a(t)}r\\
&=0.
\end{aligned}$$

The proof is completed. ■

Besides, the Lemma 7 of [13] is also useful. For the better understanding of the lemma, the proof is given here.

Lemma 4 (lemma 7 of [13]) *For the Emden equation,*

$$\begin{cases}\ddot{a}(t)=-\frac{\lambda}{a(t)},\\ a(0)=a_0>0,\ \dot{a}(0)=a_1,\end{cases} \tag{10}$$

we have, if $\lambda>0$*, there exists a finite time* $T_-<+\infty$ *such that* $a(T_-)=0$.

Proof. By integrating (10), we have

$$0\leq\frac{1}{2}\dot{a}(t)^2=-\lambda\ln a(t)+\theta \tag{11}$$

where $\theta=\lambda\ln a_0+\frac{1}{2}a_1^2$.

From (11), we get,

$$a(t)\leq e^{\theta/\lambda}.$$

If the statement is not true, we have

$$0<a(t)\leq e^{\theta/\lambda},\ \text{for all } t\geq 0.$$

But since

$$\ddot{a}(t) = -\frac{\lambda}{a(t)} \leq \frac{-\lambda}{e^{\theta/\lambda}},$$

we integrate this twice to deduce

$$a(t) \leq \int_0^t \int_0^\tau \frac{-\lambda}{e^{\theta/\lambda}} ds d\tau + C_1 t + C_0 = \frac{-\lambda t^2}{2e^{\theta/\lambda}} + C_1 t + C_0.$$

By taking t large enough, we get

$$a(t) < 0.$$

As a contradiction is met, the statement of the Lemma is true. ■

By extending the structure of the solutions (5) to the 2-dimensional isothermal Euler-Poisson equations (4) in [13], it is a natural result to get the proof of the Theorem 2.

Proof of Theorem 2. By using the Lemma 3, we can get that (7) satisfy $(6)_1$. For the momentum equation, we have,

$$\begin{aligned}
&\rho(u_t + u \cdot u_r) + K\rho_r - v(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u) \\
&= \rho\frac{\ddot{a}(t)}{a(t)}r + \frac{K}{a(t)}\rho\dot{y}(\frac{r}{a(t)}) \\
&= \frac{\rho}{a(t)}[-\frac{\lambda r}{a(t)} + K\dot{y}(\frac{r}{a(t)})].
\end{aligned}$$

By choosing

$$y(x) = \frac{\lambda}{2K}x^2 + \alpha,$$

we have verified that (7) satisfies the above $(6)_2$. If $\lambda > 0$, by the Lemma 4, there exists a finite time T for such that $a(T_-) = 0$. Thus, there exist blowup solutions in finite time T. The proof is completed. ■

With the assistance of the blowup rate results of the Euler-Poisson equations i.e. Theorem 3 in [13], it is trivial to have the following theorem:

Theorem 5 *With $\lambda > 0$, the blowup rate of the solutions (7) is,*

$$\lim_{t \to T_*} \rho(t, 0)(T_* - t)^\alpha \geq O(1),$$

where the blowup time T_* and $\alpha < N$ are constants.

Remark 6 *If we are interested in the mass of the solutions, the mass of the solutions can be calculated by:.*

$$M(t)=\int_{R^N}\rho(t,s)ds=\alpha(N)\int_0^{+\infty}\rho(t,s)s^{N-1}ds,$$

where $\alpha(N)$ denotes some constant related to the unit ball in R^N: $\alpha(1)=1$; $\alpha(2)=2\pi$; for $N\geq 3$,

$$\alpha(N)=N(N-2)V(N)=N(N-2)\frac{\pi^{N/2}}{\Gamma(N/2+1)},$$

where $V(N)$ is the volume of the unit ball in R^N and Γ is the Gamma function. We observe that the mass of the initial time 0:

(1) for $\lambda\geq 0$

$$M(0)=\frac{\alpha(N)}{a_0^N}\int_0^{+\infty}e^{\frac{\lambda}{2K}s^2+\alpha}s^{N-1}ds.$$

The mass is infinitive. The very large density comes from the ends of outside of the origin O.

(2) for $\lambda<0$,

$$M(0)=\frac{\alpha(N)}{a_0^N}\int_0^{+\infty}e^{\frac{\lambda}{2K}s^2+\alpha}s^{N-1}ds=\frac{\alpha(N)e^{\alpha}}{a_0^N}\int_0^{+\infty}e^{\frac{\lambda}{2K}s^2}s^{N-1}ds.$$

The mass of the solution can be arbitrarily small but without compact support if α is taken to be a very small negative number.

Remark 7 *Our results can be easily extended to the isothermal Euler/ Navier-Stokes equations with frictional damping term with the assistance of Lemma 7 in [12]:*

$$\begin{cases}\rho_t+u\rho_r+\rho u_r+\frac{N-1}{r}\rho u=0,\\ \rho(u_t+u\cdot u_r)+K\rho_r+\beta\rho u=v(u_{rr}+\frac{N-1}{r}u_r-\frac{N-1}{r^2}u),\end{cases}$$

where $\beta\geq 0$ and $v\geq 0$.

The solutions are:

$$\begin{cases}\rho(t,r)=\frac{e^{y(r/a(t))}}{a(t)^N},u(t,r)=\frac{\dot{a}(t)}{a(t)}r,\\ \ddot{a}(t)+\beta\dot{a}(t)=\frac{-\lambda}{a(t)},a(0)=a_0>0,\dot{a}(0)=a_1,\\ y(x)=\frac{\lambda}{2K}x^2+\alpha.\end{cases}$$

Remark 8 *Our results can be easily extended to the isothermal Euler/ Navier-Stokes equations*

with frictional damping term with the assistance of Lemma 7 in [12]:

$$\begin{cases} \rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \\ \rho(u_t + u \cdot u_r) + K\rho_r + \beta\rho u = v(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u) \end{cases}$$

where $\beta \geq 0$ *and* $v \geq 0$.

The solutions are:

$$\begin{cases} \rho(t,r) = \frac{e^{y(r/a(t))}}{a(t)^N}, u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \\ \ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda}{a(t)}, a(0) = a_0 > 0, \dot{a}(0) = a_1, \\ y(x) = \frac{\lambda}{2K}x^2 + \alpha. \end{cases}$$

Remark 9 *The solutions (5) to the Euler-Poisson equations only work for the* 2*-dimensional case. But the solutions (7) to the Navier-Stokes equations work for the* N*-dimensional* $(N \geq 1)$ *case.*

Remark 10 *We may extend the solutions to the* 2*-dimensional Euler/Navier-Stokes equations with a solid core [6]:*

$$\begin{cases} \rho_t + u\rho_r + \rho u_r + \frac{1}{r}\rho u = 0, \\ \rho\left(u_t + uu_r\right) + K\rho_r + \beta\rho u = \frac{M_0}{r} + v(u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u), \end{cases}$$

where $M_0 > 0$*, there is a unit stationary solid core locating* $[0, r_0]$*, where* r_0 *is a positive constant, surrounded by the distribution density.*

The corresponding solutions are:

$$\begin{cases} \rho(t,r) = \frac{e^{y(r/a(t))}}{a(t)^2},\ u(t,r) = \frac{\dot{a}(t)}{a(t)}r,\ \text{for } r > r_0, \\ \ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda}{a(t)},\ a(0) = a_0 > 0, \dot{a}(0) = a_1, \\ y(x) = \frac{\lambda}{2K}x^2 + M_0 \ln x + \alpha, \end{cases}$$

where $\alpha > \frac{-\lambda}{2K}$ *is a constant.*

3 Pressureless Navier-Stokes Equations with Density-dependent Viscosity

Now we consider the pressureless Navier-Stokes equations with density-dependent viscosity:

$$vis(\rho, u) \doteq \bigtriangledown(\mu(\rho) \bigtriangledown \cdot u),$$

in radial symmetry:

$$\begin{cases} \rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \\ \rho\left(u_t + uu_r\right) = (\mu(\rho))_r(\frac{N-1}{r}u + u_r) + \mu(\rho)(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u), \end{cases} \tag{12}$$

where $\mu(\rho)$ is a density-dependent viscosity function, which is usually written as $\mu(\rho) \doteq \kappa\rho^\theta$ with the constants κ, $\theta > 0$. For the study of this kind of the above system, the readers may refer [8][9][11].

We can obtain the similar estimate about Lemma 4 to the following ODE,

$$\begin{cases} \ddot{a}(t) = \frac{\lambda\dot{a}(t)}{a(t)^2}, \\ a(0) = a_0 > 0, \ \dot{a}(0) = a_1 \leq \frac{\lambda}{a_0}. \end{cases} \tag{13}$$

Lemma 11 *For the ODE (13), with $\lambda > 0$, there exists a finite time $T_- < +\infty$ such that $a(T_-) = 0$.*

Proof. (1) If $a(t) > 0$ and $\dot{a}(0) = a_1 \leq \frac{\lambda}{a_0}$ for all time t, by integrating (13), we have

$$\dot{a}(t) = -\frac{\lambda}{a(t)} - \frac{\lambda}{a_0} + a_1 \leq -\frac{\lambda}{a(t)}. \tag{14}$$

Take the integration for (14):

$$\int_0^t a(s)\dot{a}(s)ds \leq -\int_0^t \lambda ds,$$

$$\frac{1}{2}[a(t)]^2 \leq -\lambda t + \frac{1}{2}a_0^2.$$

When t is very large, we have

$$\frac{1}{2}[a(t)]^2 \leq -1.$$

A contradiction is met. The proof is completed. ■

Here we present another lemma before proceeding to the next theorem.

Lemma 12 *For the ODE*

$$\begin{cases} \dot{y}(x)y(x)^n - \xi x = 0, \\ y(0) = \alpha > 0, n \neq -1, \end{cases} \tag{15}$$

where ξ and n are constants,

we have the solution

$$y(x) = \sqrt[n+1]{\frac{1}{2}(n+1)\xi x^2 + \alpha^{n+1}}.$$

Proof. The above ODE (15) may be solved by the separation method:

$$\dot{y}(x)y(x)^n - \xi x = 0,$$

$$\dot{y}(x)y(x)^n = \xi x.$$

By taking the integration with respect to x :

$$\int_0^x \dot{y}(x)y(x)^n dx = \int_0^x \xi x dx,$$

we have,

$$\int_0^x y(x)^n d[y(x)] = \frac{1}{2}\xi x^2 + C_1, \tag{16}$$

where C_1 is a constant.

By integration by part, then the identity becomes

$$y(x)^{n+1} - n\int_0^x y(x)^{n-1}\dot{y}(x)y(x)dx = \frac{1}{2}\xi x^2 + C_1,$$
$$y(x)^{n+1} - n\int_0^x \dot{y}(x)y(x)^n dx = \frac{1}{2}\xi x^2 + C_1.$$

From the equation (16), we can have the simple expression for $y(x)$:

$$y(x)^{n+1} - n(\frac{1}{2}\xi x^2 + C_1) = \frac{1}{2}\xi x^2 + C_1,$$

$$y(x)^{n+1} = \frac{1}{2}(n+1)\xi x^2 + C_2,$$

where $C_2 = (n+1)C_1$.

By plugging into the initial condition for $y(0)$, we have

$$y(0)^{n+1} = \alpha^{n+1} = C_2.$$

Thus, the solution is:

$$y(x) = \sqrt[n+1]{\frac{1}{2}(n+1)\xi x^2 + \alpha^{n+1}}.$$

The proof is completed. ■

The family of the solution to the pressureless Navier-Stokes equations with density-dependent viscosity:

$$\left\{\begin{array}{l} \rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \\ \rho\left(u_t + uu_r\right) = (\kappa\rho^\theta)_r(\frac{N-1}{r}u + u_r) + \kappa\rho^\theta(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u), \end{array}\right. \tag{17}$$

is presented as the followings:

Theorem 13 *For the pressureless Navier-Stokes equations with density-dependent viscosity (17) in radial symmetry, there exists a family of solutions,*

for $\theta = 1$*:*

$$\begin{cases} \rho(t,r) = \frac{e^{y(r/a(t))}}{a(t)^N}, u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \\ \ddot{a}(t) = \frac{\lambda \dot{a}(t)}{a(t)^2}, a(0) = a_0 > 0, \dot{a}(0) = a_1, \\ y(x) = \frac{\lambda}{2N\kappa}x^2 + \alpha, \end{cases}$$

where α *and* λ *are arbitrary constants.*

In particular, for $\lambda > 0$ *and* $a_1 \leq \frac{\lambda}{a_0}$*, the solutions blow up in finite time;*

for $\theta \neq 1$*:*

$$\begin{cases} \rho(t,r) = \begin{cases} \frac{y(r/a(t))}{a(t)^N}, & \text{for } y(\frac{r}{a(t)}) \geq 0; \\ 0, & \text{for } y(\frac{r}{a(t)}) < 0 \end{cases}, \ u(t,r) = \frac{\dot{a}(t)}{a(t)}r, \\ \ddot{a}(t) = \frac{-\lambda \dot{a}(t)}{a(t)^{N\theta - N + 2}}, \ a(0) = a_0 > 0, \ \dot{a}(0) = a_1, \\ y(x) = \sqrt[\theta-1]{\frac{1}{2}(\theta - 1)\frac{-\lambda}{N\kappa\theta}x^2 + \alpha^{\theta-1}}, \end{cases} \tag{18}$$

where $\alpha > 0$.

Proof of Theorem 13. To $(17)_1$, we may use Lemma 3 to check it.

For $\theta = 1$, $(17)_2$, becomes:

$$\begin{aligned} &\rho(u_t + u \cdot u_r) - (\kappa\rho)_r(\frac{N-1}{r}u + u_r) - \kappa\rho_r(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u) \\ &= \rho\frac{\ddot{a}(t)}{a(t)}r - N(\frac{\kappa e^{y(r/a(t))}}{a(t)^N})_r\frac{\dot{a}(t)}{a(t)} \\ &= \rho\left(\frac{\lambda\dot{a}(t)r}{a(t)^3}\right) - \frac{N\kappa e^{y(r/a(t))}\dot{y}(\frac{r}{a(t)})}{a(t)^{N+1}}\frac{\dot{a}(t)}{a(t)} \\ &= \frac{\rho\dot{a}(t)}{a(t)^2}\left(\frac{\lambda r}{a(t)} - N\kappa\dot{y}(\frac{r}{a(t)})\right), \end{aligned} \tag{19}$$

where we use

$$\ddot{a}(t) = \frac{\lambda\dot{a}(t)}{a(t)^2}.$$

By choosing

$$y(\frac{r}{a(t)}) \doteq y(x) = \frac{\lambda}{2N\kappa}x^2 + \alpha,$$

(19) is equal to zero.

For the case of $\theta \neq 1$, $(17)_2$ can be calculated:

$$\rho(u_t + u \cdot u_r) - (\kappa\rho^\theta)_r \left(\frac{N-1}{r}u + u_r\right) - \kappa\rho^\theta(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u) \tag{20}$$

$$= \rho\left(-\frac{\lambda\dot{a}(t)r}{a(t)^{N\theta-N+2}a(t)}\right) - \frac{N\kappa\theta y(\frac{r}{a(t)})^{\theta-1}\dot{y}(\frac{r}{a(t)})}{a(t)^{N(\theta-1)}a(t)^{N+1}}\frac{\dot{a}(t)}{a(t)}$$

$$= \rho\left(-\frac{\lambda\dot{a}(t)r}{a(t)^{N\theta-N+2}a(t)}\right) - \frac{N\kappa\theta y(\frac{r}{a(t)})y(\frac{r}{a(t)})^{\theta-2}\dot{y}(\frac{r}{a(t)})\dot{a}(t)}{a(t)^N a(t)^{N\theta-N+2}}$$

$$= \rho\left(-\frac{\lambda\dot{a}(t)r}{a(t)^{N\theta-N+2}a(t)}\right) - \frac{N\kappa\theta\rho y(\frac{r}{a(t)})^{\theta-2}\dot{y}(\frac{r}{a(t)})\dot{a}(t)}{a(t)^{N\theta-N+2}}$$

$$= \frac{-\rho\dot{a}(t)}{a(t)^{N\theta-N+2}}\left(-\frac{\lambda r}{a(t)} + N\kappa\theta y(\frac{r}{a(t)})^{\theta-2}\dot{y}(\frac{r}{a(t)})\right). \tag{21}$$

Define $x \doteq \frac{r}{a(t)}$, $n \doteq \theta - 2$, it follows:

$$= \frac{-\rho\dot{a}(t)}{a(t)^{N\theta-N+2}}\left(\lambda x + N\kappa\theta y(x)^n\dot{y}(x)\right) \tag{22}$$

$$= \frac{-\lambda\rho\dot{a}(t)}{a(t)^{N\theta-N+2}}\left(x + \frac{N\kappa\theta}{\lambda}y(x)^n\dot{y}(x)\right), \tag{23}$$

and $\xi \doteq \frac{\lambda}{N\kappa\theta}$ in Lemma 12, and choose

$$y(\frac{r}{a(t)}) \doteq y(x) = \sqrt[\theta-1]{\frac{1}{2}(\theta-1)\frac{-\lambda}{N\kappa\theta}x^2 + \alpha^{\theta-1}}.$$

And this is easy to check that

$$\dot{y}(0) = 0.$$

The equation (22) is equal to zero. The proof is completed. ■

Remark 14 *By controlling the initial conditions in some solutions (18), we may get the blowup solutions. And the modified solutions can be extended to the system in radial symmetry with frictional damping:*

$$\begin{cases} \rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0, \\ \rho\left(u_t + uu_r\right) + \beta\rho u = (\mu(\rho))_r\left(\frac{N-1}{r}u + u_r\right) + \mu(\rho)(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u), \end{cases}$$

where $\beta > 0$,

with the assistance of the ODE:

$$\begin{cases} \ddot{a}(t) + \beta\dot{a}(t) = \frac{-\lambda\dot{a}(t)}{a(t)^S}, \\ a(0) = a_0 > 0, \dot{a}(0) = a_1, \end{cases}$$

where S is a constant.

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